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Bounded Area Tests For Comparing The Dynamics Between ARMA Processes

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This article presents a new test for discerning whether or not two independent autoregressive moving average (ARMA) processes have the same autocovariance structure. This test utilizes a specific geometric feature of a time series plot, namely the area bounded between the line segments that connect adjacent points and the time axis. It will be shown that if you sample two ARMA processes and calculate the magnitudes of the two resulting bounded areas, then a significant difference among these areas tends to imply a significant difference in autocovariances.

Keywords Bounded area; Autocovariance; ARMA.

Mathematics Subject Classification Primary 62M10; Secondary 60G10.

1. Introduction

Consider the causal stationary ARMA(p, q) time series $\{X_t\}$ defined by

$$X_t - \sum_{i=1}^{p} \phi_i X_{t-i} = \epsilon_t + \sum_{j=1}^{q} \theta_j \epsilon_{t-j} \qquad t = 0, \pm 1, \pm 2, \ldots, \tag{1.1}$$

where $\mathrm{E}(X_t) = 0$, $\mathrm{E}(X_t^2) < \infty$, $\{\epsilon_t\} \sim \mathrm{IID}(0, \sigma^2)$, and the roots of $\phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p$ fall outside the unit circle. $\{X_t\}$ then has a linear process representation given by

$$X_t = \sum_{j=0}^{\infty} \psi_j \epsilon_{t-j} ,$$

where the ψ_j's are absolutely summable, and an autocovariance function given by

$$\gamma_X(h) = \mathrm{Cov}(X_t, X_{t+h}) = \sigma^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+h} \qquad h = 0, 1, 2, \ldots .$$

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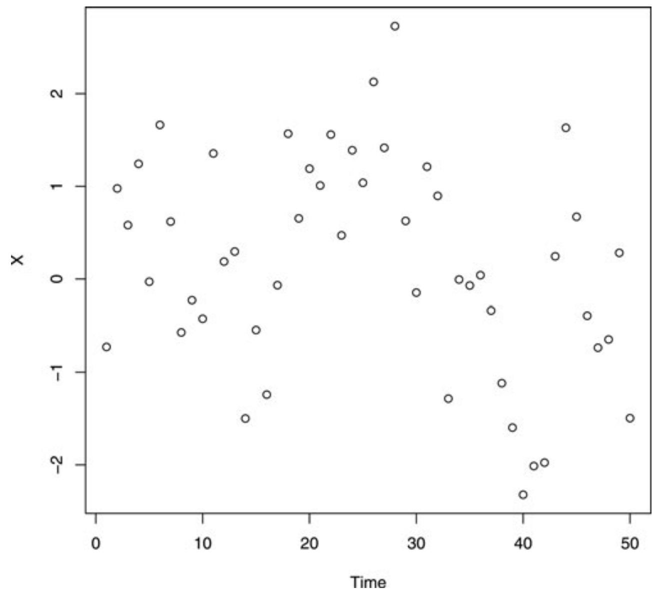


Figure 1. Scatterplot of an arbitrary mean-zero series.

The goal of this article is to compare the autocovariances of two independent ARMA series. That is, if $\{X_t\}$ and $\{Y_t\}$ each follow (1.1) and are independent of one another, we specifically wish to test

$$H_0 : \gamma_X(h) = \gamma_Y(h) \text{ for all } h \qquad \text{vs.}$$
$$H_1 : \gamma_X(h) \neq \gamma_Y(h) \text{ for at least one } h. \tag{1.2}$$

In general, devising tests to compare the dynamics between two time series is nothing new. A wide variety of such tests exist, many of which are outlined in the thorough survey found in Caiado et al. (2006). The test developed in this article calls upon a simple, heretofore overlooked, geometric feature of a time series plot that we henceforward refer to as the *bounded area*. As will be seen, the mean zero assumption in (1.1) is absolutely necessary and so if an ARMA series has an intercept term, then the series must be demeaned beforehand.

Observe the scatterplot of an arbitrary mean-zero series shown in Fig. 1. Typically, one "connects the dots" with arc length segments for better presentation, as shown in Fig. 2. The shaded region between these segments and the time axis, as shown in Fig. 3, is the bounded area referred to above and will be used to test (1.2). It will be shown that if two independent processes from (1.1) are observed over the same time period, then a significant difference in bounded area *magnitudes* implies a significant difference in autocovariance structures.

For example, Fig. 4 shows the bounded areas for sample realizations of two AR(1) processes ($\phi = -0.9$ and $\phi = 0.9$), each with i.i.d. standard normal errors, when $t = 1, 2, \ldots, 50$. The former has a bounded area of 46.94621 square units while the latter has a bounded area of 52.13977 square units. We know the autocovariance functions associated with these two series are significantly different, but how do we know if the two

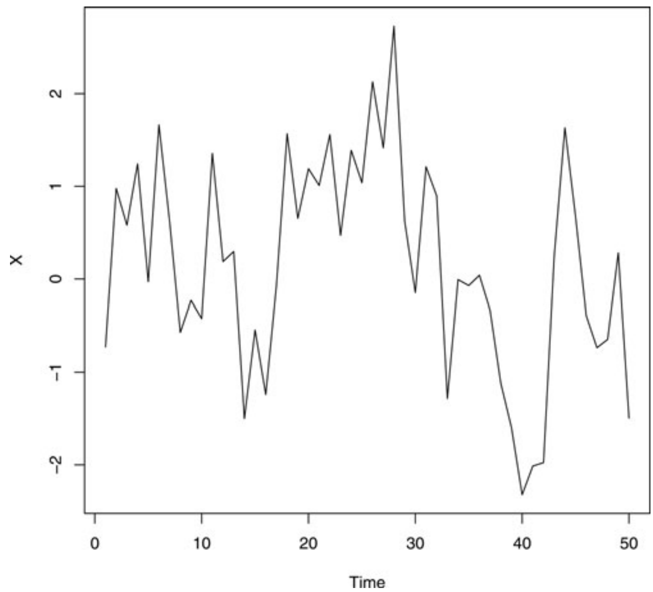


Figure 2. Scatterplot of series from Figure 1 now with arc length segments connecting adjacent points.

bounded areas are significantly different? Section 2 answers this type of question with the development of a meaningful way to compare such values (i.e., a bounded area test). In Sec. 3, we review the autocovariance equality tests devised by Lund et al. (2009) and Tunno et al. (2012) and then compare them with the bounded area test with various simulations. Section 4 concludes the article with some remarks.

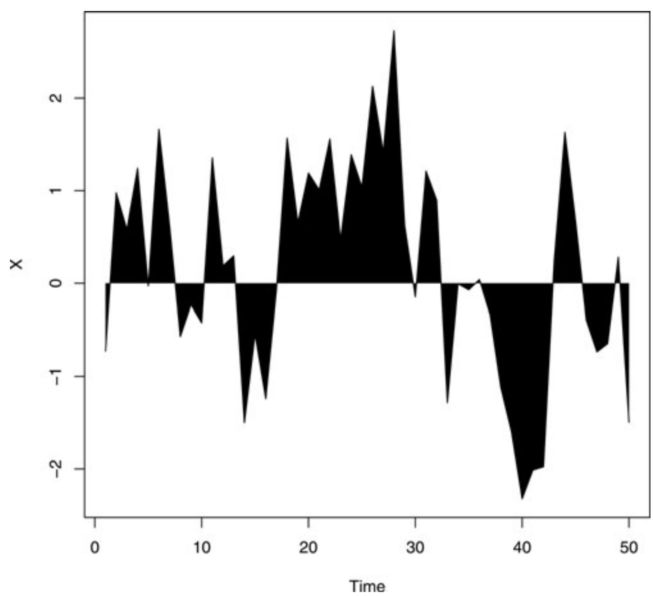


Figure 3. Scatterplot of series from Figures 1 and 2 now with shading indicating the bounded area between arc length segments and time axis.

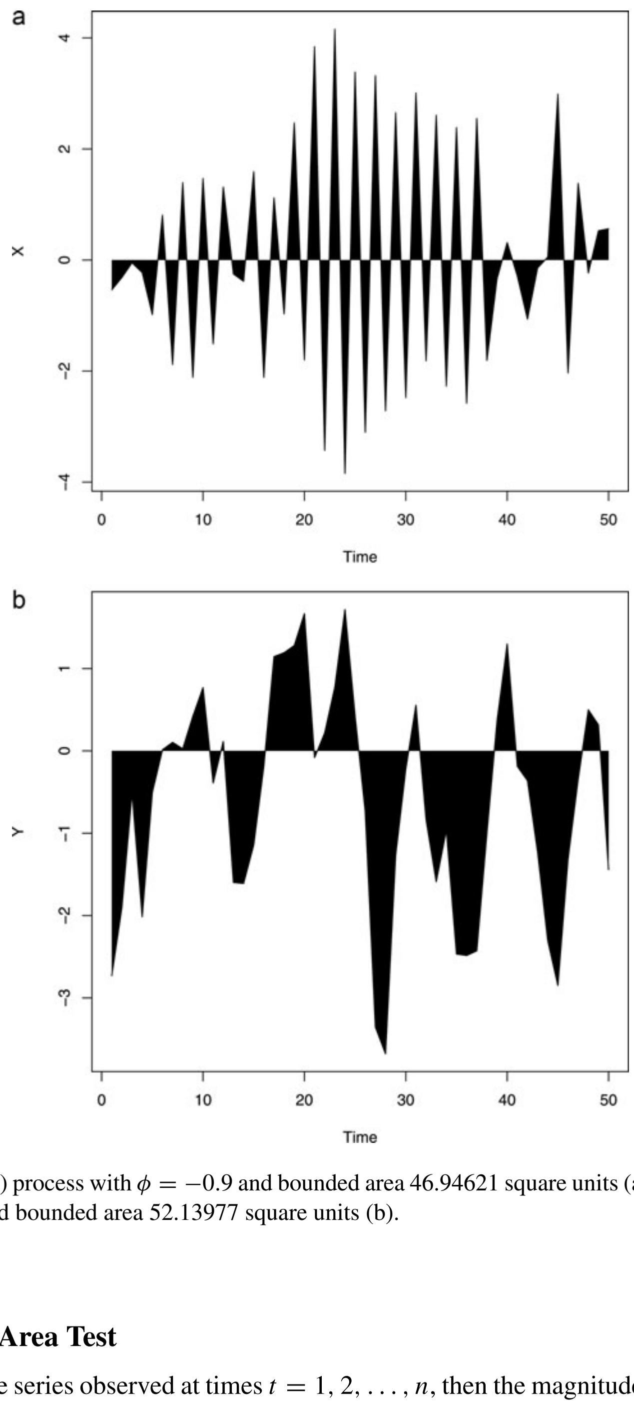


Figure 4. AR(1) process with $\phi = -0.9$ and bounded area 46.94621 square units (a). AR(1) process with $\phi = 0.9$ and bounded area 52.13977 square units (b).

2. Bounded Area Test

If $\{X_t\}$ is a time series observed at times $t = 1, 2, \ldots, n$, then the magnitude of its bounded area is equal to

$$A_n^X = \sum_{t=2}^{n} \left[\int_{t-1}^{t} \left| (X_t - X_{t-1})(u - t) + X_t \right| du \right] = \sum_{t=2}^{n} I_t^X, \tag{2.3}$$

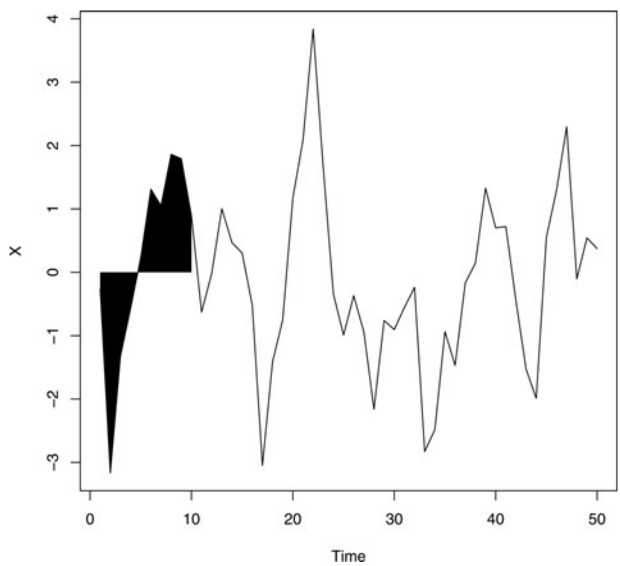


Figure 5. Arbitrary mean-zero series with shaded region corresponding to A_{10}^X.

where I_w^X is the magnitude from $t = w - 1$ to $t = w$. To illustrate, Figs. 5, 6, and 7 show an arbitrary mean-zero time series observed over $t = 1, 2, \ldots, 50$. The shaded regions in these figures correspond to A_{10}^X, A_{30}^X, and A_{50}^X, respectively.

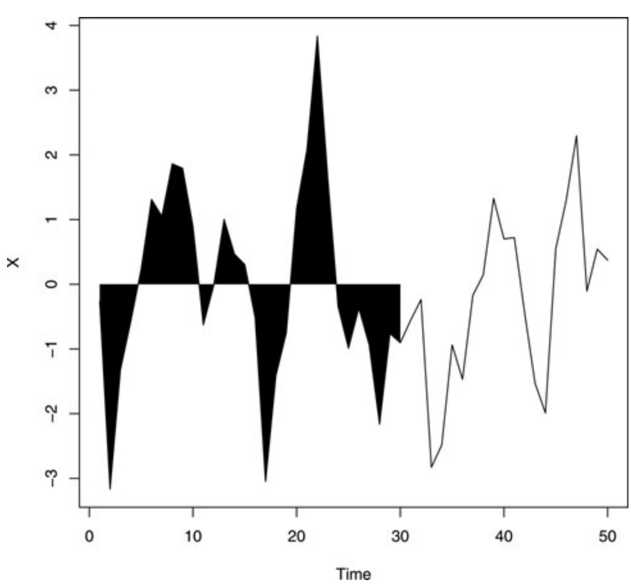


Figure 6. Arbitrary mean-zero series with shaded region corresponding to A_{30}^X.

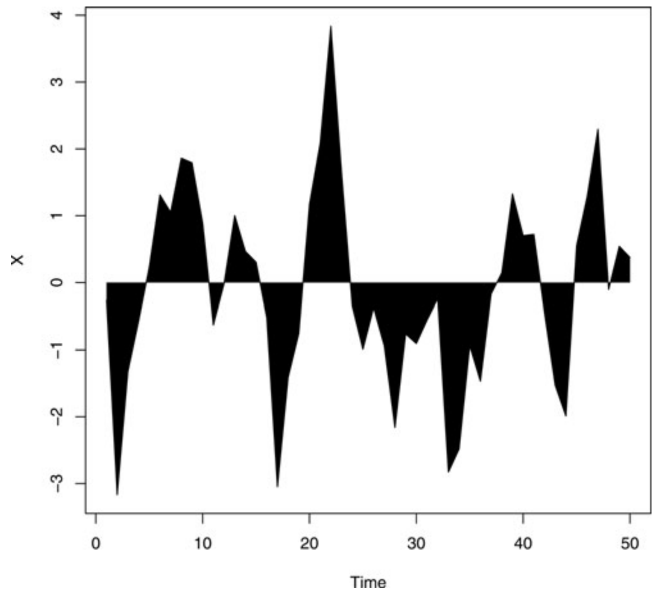


Figure 7. Arbitrary mean-zero series with shaded region corresponding to A_{50}^X.

Now assume that $\{X_t\}$ follows (1.1). Since $\{X_t\}$ is a stationary process (in the wide sense), so is $\{I_t^X\}$. Thus, we have

$$\text{Var}\left(A_n^X\right) = \text{Var}\left(\sum_{t=2}^{n} I_t^X\right) = (n-1)\gamma_{I^X}(0) + 2\sum_{h=1}^{n-2}(n-1-h)\gamma_{I^X}(h),$$

where $\gamma_{I^X}(h) = \text{Cov}(I_t^X, I_{t+h}^X)$. If $\{Y_t\}$ also follows (1.1), then A_n^Y and its associated functionals are defined analogously.

If $\{X_t\}$ and $\{Y_t\}$ are independent of one another, then given samples $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_n$, a normalized test statistic for (1.2) is equal to

$$T = \frac{A_n^X - A_n^Y - \text{E}\left(A_n^X - A_n^Y\right)}{\sqrt{\text{Var}\left(A_n^X - A_n^Y\right)}} = \frac{A_n^X - A_n^Y - \left(\text{E}\left(A_n^X\right) - \text{E}\left(A_n^Y\right)\right)}{\sqrt{\text{Var}\left(A_n^X\right) + \text{Var}\left(A_n^Y\right)}}.$$

Under the assumption that the null hypothesis in (1.2) implies $\text{E}(A_n^X) = \text{E}(A_n^Y)$, we then have

$$T \overset{H_0}{=} \frac{A_n^X - A_n^Y}{\sqrt{\text{Var}\left(A_n^X\right) + \text{Var}\left(A_n^Y\right)}}. \tag{2.4}$$

Theorem 2.1. *Let $\{X_t\}$ and $\{Y_t\}$ each follow model (1.1) and be independent of one another. Then, under the assumption of the null hypothesis in (1.2), the test statistic in (2.4) converges in distribution to the standard normal. That is, $T \xrightarrow{D} N(0,1)$ when H_0 is true.*

Proof. We will use Corollary 1 of Theorem 1 from Wu (2002). We begin with a list of all relevant objects concerning sample $X_1, X_2, \dots, X_n$:

(A) $X_n = \sum_{i=0}^{\infty} \psi_i \epsilon_{n-i}$

(B) $X_{n,-} = \sum_{i=n}^{\infty} \psi_i \epsilon_{n-i}$

(C) $I_n^X = \int_{n-1}^{n} \Big|(X_n - X_{n-1})(u-n) + X_n\Big|\, du$

(D) $I_{n,-}^X = \int_{n-1}^{n} \Big|(X_{n,-} - X_{n-1,-})(u-n) + X_{n,-}\Big|\, du$

(E) $K(X_{n-1}, X_n) = I_n^X - \mathrm{E}\left(I_n^X\right)$

(F) $K(X_{n-1,-}\,,\, X_{n,-}) = I_{n,-}^X - \mathrm{E}\left(I_{n,-}^X\right)$

(A) follows from the causality of (1.1), while (B) is a truncated version of (A). Similarly, (D) and (F) are truncated versions of (C) and (E), respectively.

Now, observe that

$$
\begin{aligned}
&\mathrm{E}(K(X_{n-1,-}\,,\, X_{n,-}))^2 \ \le\ \mathrm{E}\left(I_{n,-}^X\right)^2 \\
&\le\ \mathrm{E}\left(\int_{n-1}^{n} \Big|(X_{n,-} - X_{n-1,-})(u-n)\Big|\, du + \int_{n-1}^{n} |X_{n,-}|\, du \right)^2 \\
&=\ \mathrm{E}\left(\Big|X_{n,-} - X_{n-1,-}\Big| \int_{n-1}^{n} (n-u)\, du + |X_{n,-}| \int_{n-1}^{n} du \right)^2 \\
&=\ \mathrm{E}\left(\frac{|X_{n,-} - X_{n-1,-}|}{2} + |X_{n,-}| \right)^2 \ \le\ v\mathrm{E}(X_{n,-})^2
\end{aligned}
$$

for some constant $v > 0$ since $|X_{n,-} - X_{n-1,-}| < \infty$. If $||\cdot||$ denotes the $\mathcal{L}^2$-norm, it follows that

$$
\begin{aligned}
\sum_{n=2}^{\infty} \frac{||K(X_{n-1,-}\,,\, X_{n,-})||}{\sqrt{n}} \ &\le\ v^{1/2} \sum_{n=2}^{\infty} \frac{||X_{n,-}||}{\sqrt{n}} \\
&=\ v^{1/2} \sum_{n=2}^{\infty} \sqrt{\mathrm{E}\left(\sum_{i=n}^{\infty} \psi_i \epsilon_{n-i}\right)^2 \Big/ n} \\
&\le\ v^{1/2} \sum_{n=2}^{\infty} \sqrt{\mathrm{E}\left(\sum_{i=n}^{\infty} |\psi_i| \sqrt{n}\right)^2 \Big/ n} \\
&=\ v^{1/2} \sum_{n=2}^{\infty} \sum_{i=n}^{\infty} |\psi_i| \ <\ \infty,
\end{aligned}
$$

since for any causal ARMA process, we have $|\psi_k| \leq Cr^k$ for some $C \in \mathbb{R}^+$ and $r \in (0, 1)$. Then, by Corollary 1 of Theorem 1 from Wu (2002), we have

$$L_K^X = \frac{1}{\sqrt{n}} \sum_{t=2}^{n} K(X_{t-1}, X_t) = \frac{A_n^X - \mathrm{E}\left(A_n^X\right)}{\sqrt{n}} \xrightarrow{D} N\left(0, \sigma_K^2\right),$$

where $\sigma_K^2 = \lim_{n\to\infty} \mathrm{Var}(L_K^X)$. An analogous result holds for $\{Y_t\}$. Thus, we have

$$\frac{L_K^X - L_K^Y}{\sqrt{\mathrm{Var}\left(L_K^X - L_K^Y\right)}} \stackrel{H_0}{=} \frac{A_n^X - A_n^Y}{\sqrt{\mathrm{Var}\left(A_n^X\right) + \mathrm{Var}\left(A_n^Y\right)}} \xrightarrow{D} N(0, 1)$$

by Slutsky's theorem and the independence of $\{X_t\}$ and $\{Y_t\}$. □

Theorem 2.1 gives us a meaningful way to compare the bounded area magnitudes for two independent series that both follow (1.1). As will be shown in the next section, a significant difference in bounded area magnitudes implies a significant difference in autocovariance structures. Thus, the *bounded area test of size* α tells us to reject the null hypothesis in (1.2) if the magnitude of our test statistic in (2.4) exceeds $z_{\alpha/2}$, where $z_{\alpha/2}$ is the standard normal critical value with area $\alpha/2$ to its right. That is, we reject H_0 if $|T| > z_{\alpha/2}$.

It should be noted that in practice $\gamma_{I^X}(h)$ is unknown, but $\mathrm{Var}(A_n^X)$ can still be estimated with

$$\widehat{\mathrm{Var}}\left(\sum_{t=2}^{n} I_t^X\right) = (n-1)\widehat{\gamma}_{I^X}(0) + 2\sum_{h=1}^{[\sqrt[3]{n}]}(n-1-h)\widehat{\gamma}_{I^X}(h), \qquad (2.5)$$

where the non negative definite autocovariance estimator

$$\widehat{\gamma}_{I^X}(h) = \frac{1}{n-1}\sum_{t=2}^{n-h}\left(I_t^X - \bar{I}\right)\left(I_{t+h}^X - \bar{I}\right) \qquad h = 0, 1, 2, \ldots, n-2$$

is used with

$$\bar{I} = \frac{1}{n-1}\sum_{t=2}^{n} I_t^X = \frac{A_n^X}{n-1}.$$

The sum in (2.5) is truncated at the greatest integer less than or equal to $\sqrt[3]{n}$ in order to avoid any bias associated with large lags (Berkes et al. 2009, discuss this and other truncation schemes). (2.5) is also a consistent estimator and so if it and its $\{Y_t\}$ analog replace $\mathrm{Var}(A_n^X)$ and $\mathrm{Var}(A_n^Y)$ in the denominator of (2.4), Theorem 2.1 still holds.

3. Simulations

3.1. Set-up

This section compares the Type I error and power of the bounded area test, the time domain test of Lund et al. (2009), and the arc length test of Tunno et al. (2012) for testing (1.2) at level $\alpha = 0.05$. In all figures, these tests will be abbreviated as AR, TD, and AL, respectively.

For each figure, series of length $n = 1,000$ were generated while 10,000 independent simulations were conducted to estimate the error and power values. $\{X_t\}$ and $\{Y_t\}$ are independent of each other and both follow (1.1). All error terms are i.i.d. standard normal.

While the bounded area test only requires a series to have a finite second moment, the other two tests require a finite fourth moment, but since our simulations use normal errors, moments of all orders will be finite. Below is a brief summary of the time domain and arc length tests.

3.2. Time Domain and Arc Length Tests

The time domain test of Lund et al. (2009) is based on a result from Bartlett (see Ch. 7 of Brockwell and Davis, 2006) which states that

$$\begin{bmatrix} \widehat{\gamma}(0) \\ \widehat{\gamma}(1) \\ \widehat{\gamma}(2) \\ \vdots \\ \widehat{\gamma}(L) \end{bmatrix} \sim AN\left(\begin{bmatrix} \gamma(0) \\ \gamma(1) \\ \gamma(2) \\ \vdots \\ \gamma(L) \end{bmatrix}, \frac{\mathbf{W}}{n} \right)$$

as $n \to \infty$, where $\mathbf{W}$ is an $(L+1) \times (L+1)$ matrix with $(i, j)^{th}$ entry

$$W_{ij} = (\eta - 3)\gamma(i)\gamma(j) + \sum_{k=-\infty}^{\infty} [\gamma(k)\gamma(k-i+j) + \gamma(k+j)\gamma(k-i)]$$

for $0 \le i, j \le L$.Under the null hypothesis of (1.2),

$$V = \begin{bmatrix} \widehat{\gamma}_X(0) - \widehat{\gamma}_Y(0) \\ \widehat{\gamma}_X(1) - \widehat{\gamma}_Y(1) \\ \widehat{\gamma}_X(2) - \widehat{\gamma}_Y(2) \\ \vdots \\ \widehat{\gamma}_X(L) - \widehat{\gamma}_Y(L) \end{bmatrix} \sim AN\left(\mathbf{0}, \frac{2\mathbf{W}}{n}\right)$$

as $n \to \infty$, where $\widehat{\gamma}_X(h)$ and $\widehat{\gamma}_Y(h)$ estimate lag h autocovariances of $\{X_t\}$ and $\{Y_t\}$, respectively. It follows that

$$\left(\frac{n}{2}\right) V^T \mathbf{W}^{-1} V \overset{D}{\to} \chi^2_{L+1},$$

where $\mathbf{W}$ is estimated by $\widehat{\mathbf{W}}$. This estimate replaces $\gamma(h)$ with $(\widehat{\gamma}_X(h) + \widehat{\gamma}_Y(h))/2$ and truncates the infinite sum with index values between $\pm[\sqrt[3]{n}]$ (again, see Berkes et al. 2009). Thus, if $(n/2)V^T\widehat{\mathbf{W}}^{-1}V$ exceeds the $(1-\alpha)^{\text{th}}$ quantile of the χ^2_{L+1} distribution, the null hypothesis is rejected. For the simulations below, $L = 10$ with critical value $\chi^2_{11,0.05} = 19.68$.

The arc length test of Tunno et al. (2012) is very similar to the bounded area test in that it is centered around a specific geometric feature of a time series plot. The arc length of time series $\{X_t\}$ observed over $t = 1, 2, \ldots, n$ is equal to

$$\sum_{t=2}^{n} \sqrt{1 + (X_t - X_{t-1})^2} = \sum_{t=2}^{n} S_t^X$$

and is simply the sum of the lengths of the $n-1$ line segments connecting adjacent points on the scatter plot (see Figs. 1 and 2). Since $\{S_t^X\}$ is stationary, we have

$$\mathrm{Var}\left(\sum_{t=2}^{n} S_t^X\right) = (n-1)\gamma_{S^X}(0) + 2\sum_{h=1}^{n-2}(n-1-h)\gamma_{S^X}(h),$$

where $\gamma_{S^X}(h) = \mathrm{Cov}(S_t^X, S_{t+h}^X)$. All objects are defined analogously for process $\{Y_t\}$.

If the arc lengths of $\{X_t\}$ and $\{Y_t\}$ are significantly different, then their autocovariance structures tend to be as well. The comparison process is made rigorous by utilizing the normalized test statistic

$$U = \frac{\left(\sum_{t=2}^{n} S_t^X - \sum_{t=2}^{n} S_t^Y\right) - (n-1)\left(\mathrm{E}\left(S_t^X\right) - \mathrm{E}\left(S_t^Y\right)\right)}{\sqrt{\mathrm{Var}\left(\sum_{t=2}^{n} S_t^X - \sum_{t=2}^{n} S_t^Y\right)}}.$$

Under the assumption that the null hypothesis in (1.2) implies $\mathrm{E}(S_t^X) = \mathrm{E}(S_t^Y)$, and that $\{X_t\}$ and $\{Y_t\}$ are independent of one another, we then have

$$U \overset{H_0}{=} \frac{\sum_{t=2}^{n} S_t^X - \sum_{t=2}^{n} S_t^Y}{\sqrt{\mathrm{Var}\left(\sum_{t=2}^{n} S_t^X\right) + \mathrm{Var}\left(\sum_{t=2}^{n} S_t^Y\right)}} \overset{D}{\to} N(0,1).$$

Thus, the arc length test of size α rejects the null hypothesis of (1.2) if $|U| > z_{\alpha/2}$, where $z_{\alpha/2}$ is the standard normal critical value with area $\alpha/2$ to its right. Consistent estimators of the variances can be used as surrogates without affecting the test.

3.3. Simulations

In Fig. 8a, we look at the Type I error values when $\{X_t\}$ and $\{Y_t\}$ both follow an MA(1) model with $4 \le \theta_X = \theta_Y \le 6$. All three tests have a low overall error, especially the time domain test. In Fig. 8b, we look at the power values when both processes follow an MA(1) model with $\theta_X = 5$ and $4 \le \theta_Y \le 6$. All three tests increase power as the magnitude of θ_Y deviates from $\theta_X = 5$.

In Fig. 9a, we look at the error values when $\{X_t\}$ and $\{Y_t\}$ both follow an MA(1) model with $-6 \le \theta_X = \theta_Y \le -4$. All three tests have a low overall error, especially the time domain test. In Fig. 9b, we look at the power values when both processes follow an MA(1) model with $\theta_X = -5$ and $-6 \le \theta_Y \le -4$. All three tests increase power as θ_Y deviates from $\theta_X = -5$, especially the bounded area test.

In Fig. 10a, we look at the error values when $\{X_t\}$ and $\{Y_t\}$ both follow an MA(2) model with $\theta_2^X = \theta_2^Y = 5$ and $-6 \le \theta_1^X = \theta_1^Y \le -4$. Arc length and bounded area have a low error, but the time domain test has a slightly lower error error for parameter values close to -4 and a slightly higher error for parameter values close to -6. In Fig. 10b, we look at the power values when both processes follow an MA(2) model with $\theta_2^X = \theta_2^Y = 5$, $\theta_1^X = -5$, and $-6 \le \theta_1^Y \le -4$. All three tests increase power as θ_1^Y deviates from $\theta_1^X = -5$, especially the time domain test, although its power may be slightly inflated for parameter values close to -6.

In Fig. 11, we look at the error values when $\{X_t\}$ and $\{Y_t\}$ both follow an AR(1) model with $-1 < \phi_X = \phi_Y < 1$. All three tests have a low overall error, especially the time domain test. When the parameter has magnitude near 1 (which is where stationarity breaks down), the error of the bounded area and time domain tests blow up, especially the former.

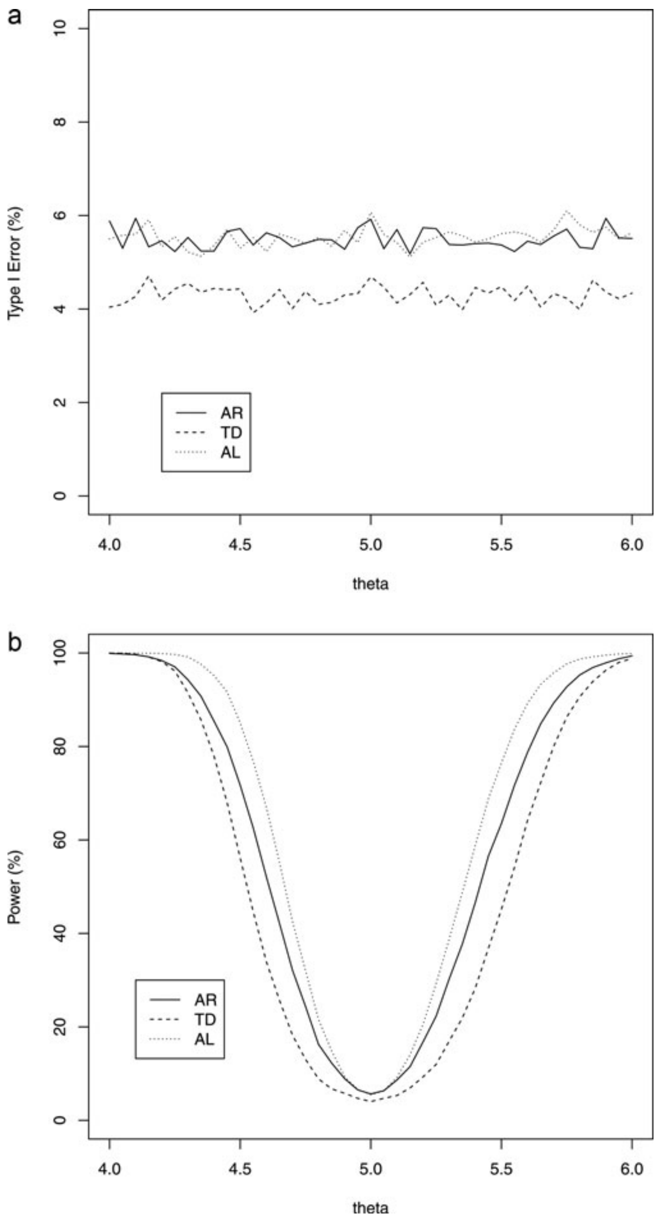


Figure 8. (a) Type I error when $\{X_t\}$ and $\{Y_t\}$ both follow an MA(1) model with $4 \leq \theta_X = \theta_Y \leq 6$. (b) Power when $\{X_t\}$ and $\{Y_t\}$ both follow an MA(1) model with $\theta_X = 5$ and $4 \leq \theta_Y \leq 6$.

This is also true for the arc length test when the parameter is near -1, but its error remains low when the parameter is near 1.

In Fig. 12a, we look at the power values when $\{X_t\}$ and $\{Y_t\}$ both follow an AR(1) model with $\phi_X = 0.5$ and $0.05 \leq \phi_Y \leq 0.95$. All three tests increase power as ϕ_Y deviates from $\phi_X = 0.5$, especially the bounded area and time domain tests.

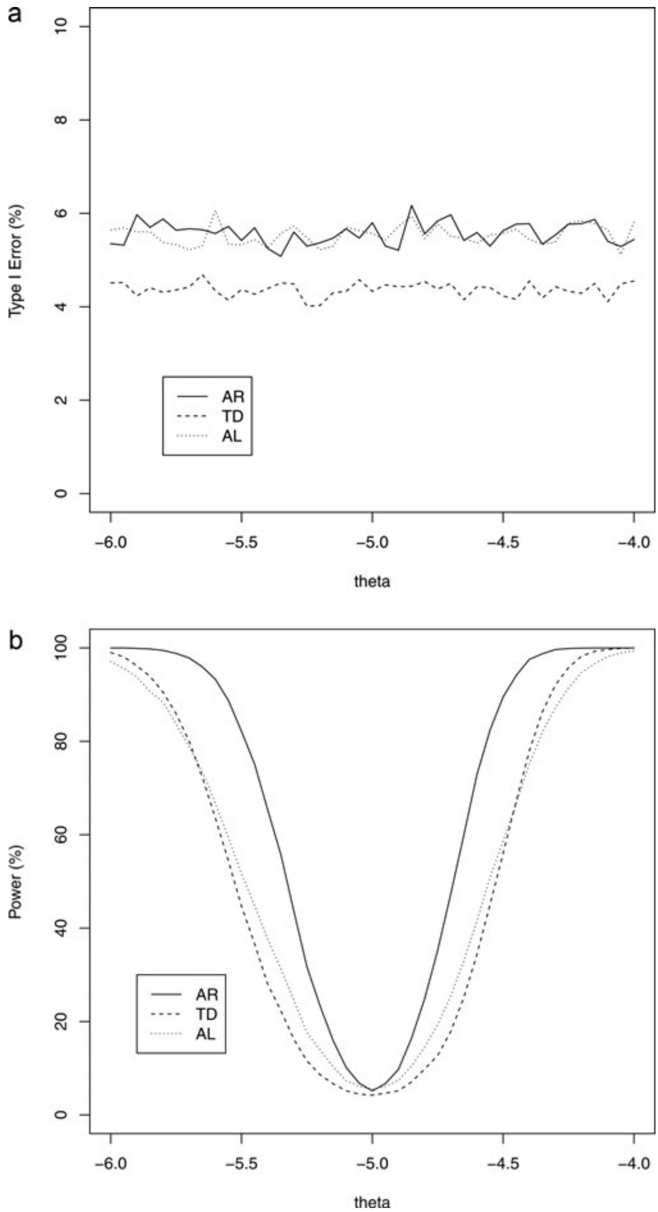


Figure 9. (a) Type I error when $\{X_t\}$ and $\{Y_t\}$ both follow an MA(1) model with $-6 \leq \theta_X = \theta_Y \leq -4$. (b) Power when $\{X_t\}$ and $\{Y_t\}$ both follow an MA(1) model with $\theta_X = -5$ and $-6 \leq \theta_Y \leq -4$.

In Fig. 12b, we look at the power values when $\{X_t\}$ and $\{Y_t\}$ both follow an AR(1) model with $\phi_X = -0.5$ and $-0.95 \leq \phi_Y \leq -0.05$. The arc length and time domain tests increase power as ϕ_Y deviates from $\phi_X = -0.5$, but the bounded area test has extremely low power for $-0.5 \leq \phi_Y \leq -0.05$.

In Fig. 13a, we look at the error values when $\{X_t\}$ and $\{Y_t\}$ both follow an AR(2) model with $\phi_2^X = \phi_2^Y = -0.5$ and $0 \leq \phi_1^X = \phi_1^Y \leq 1$. All three tests have a low overall

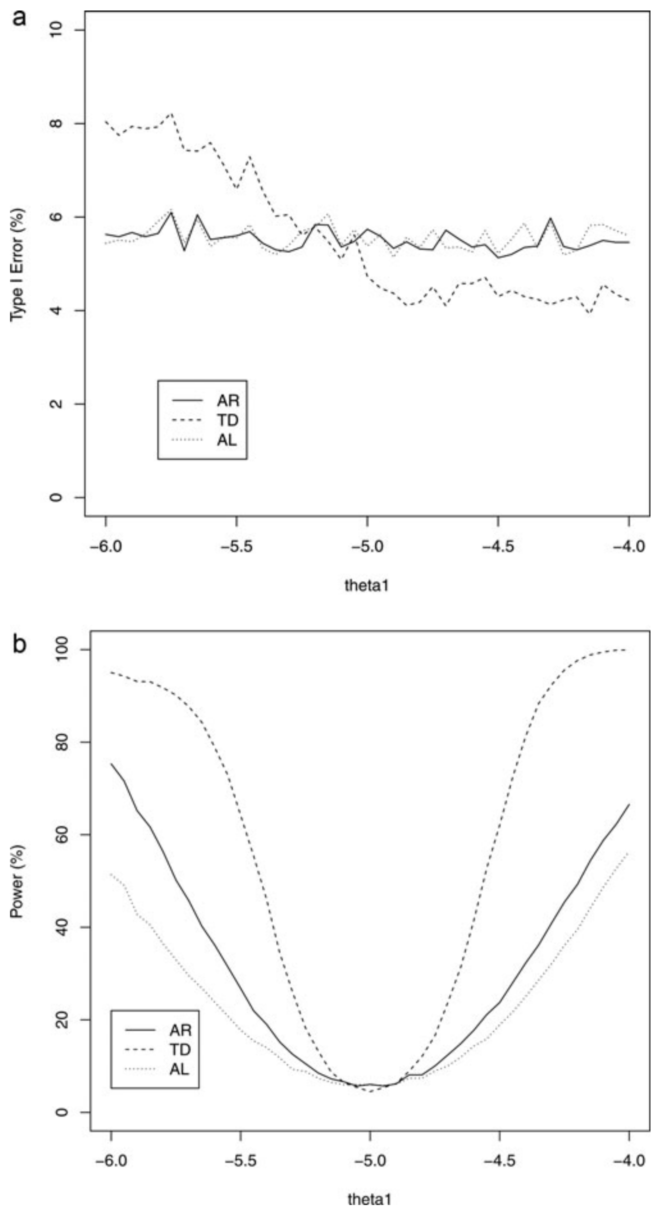


Figure 10. (a) Type I error when $\{X_t\}$ and $\{Y_t\}$ both follow an MA(2) model with $\theta_2^X = \theta_2^Y = 5$ and $-6 \leq \theta_1^X = \theta_1^Y \leq -4$. (b) Power when $\{X_t\}$ and $\{Y_t\}$ both follow an MA(2) model with $\theta_2^X = \theta_2^Y = 5$, $\theta_1^X = -5$, and $-6 \leq \theta_1^Y \leq -4$.

error, especially the time domain test. In Fig. 13b, we look at the power values when $\{X_t\}$ and $\{Y_t\}$ both follow an AR(2) model with $\phi_2^X = \phi_2^Y = -0.5$, $\phi_1^X = 0.5$ and $0 \leq \phi_1^Y \leq 1$. All three tests increase power as ϕ_1^Y deviates from $\phi_1^X = 0.5$, especially the time domain test.

In Fig. 14a, we look at the error values when $\{X_t\}$ and $\{Y_t\}$ both follow an ARMA(1,1) model with $\theta_X = \theta_Y = 2$ and $-0.5 \leq \phi_X = \phi_Y \leq 0.5$. All three tests have a low overall

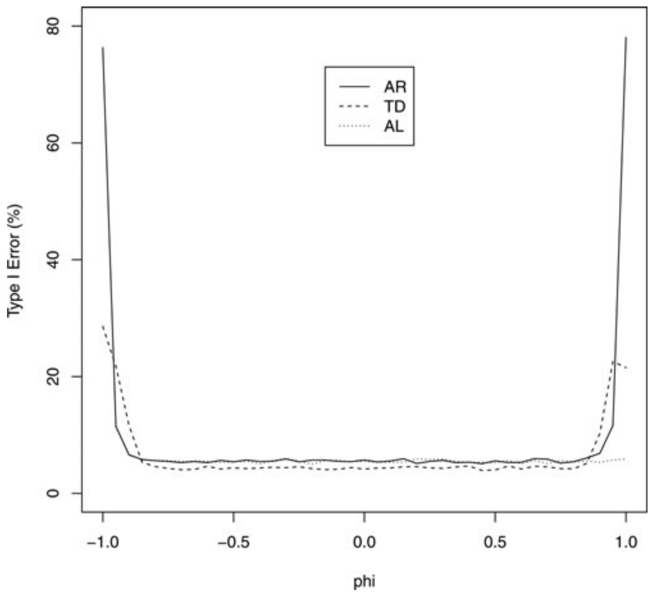


Figure 11. Type I error when $\{X_t\}$ and $\{Y_t\}$ both follow an AR(1) model with $-1 < \phi_X = \phi_Y < 1$.

error, especially the time domain test. In Fig. 14b, we look at the power values when $\{X_t\}$ and $\{Y_t\}$ both follow an ARMA(1,1) model with $\theta_X = \theta_Y = 2$, $\phi_X = 0$ and $-0.5 \leq \phi_Y \leq 0.5$. All three tests increase power as ϕ_Y deviates from $\phi_X = 0$, especially the bounded area and time domain tests. The arc length test has particularly weak power when $\phi_Y > 0$.

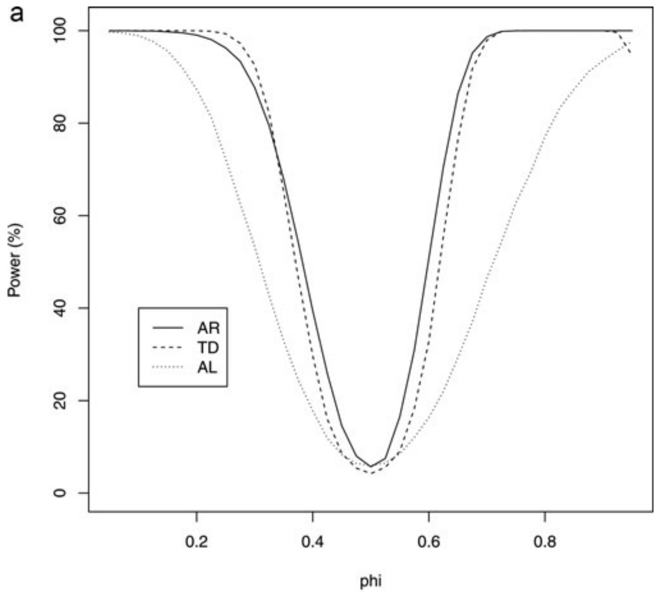
4. Closing Remarks

The simulations in the previous section reveal the bounded area test to be a reasonable, albeit imperfect, one. Overall, it competes respectably with both the time domain and arc length tests with one main exception, of course, coming from Fig. 12b. In this case, we see that the bounded area test has problems distinguishing between two AR(1) processes where one parameter is fixed at -0.5 and the other varies between -0.95 and -0.05.

Fig. 15a shows the power when $\{X_t\}$ and $\{Y_t\}$ both follow an AR(1) model with $\phi_X = -0.05$ and $-0.95 \leq \phi_Y \leq -0.05$. Once again, bounded area power is weak. On the other hand, Fig. 15b shows the power when $\{X_t\}$ and $\{Y_t\}$ both follow an AR(1) model with $\phi_X = -0.95$ and $-0.95 \leq \phi_Y \leq -0.05$. Here, the bounded area test is extremely powerful. Thus, it appears that problems arise not simply when the parameters are negative, but when they are both close to zero as well. Further investigation is currently underway.

Aside from the AR(1) anomaly, the bounded area test is clearly picking up differences between autocovariance structures. Further confirmation of this fact appears in the next figure. Consider the case where $\{X_t\}$ and $\{Y_t\}$ both follow an MA(1) model with parameters $\theta_X = \sqrt{2}$ and $\theta_Y = 1/\sqrt{2}$ and error variances $\sigma_X^2 = 1$ and $\sigma_Y^2 = 2$. Here we have two different series that have the same autocovariance function. Specifically,

$$\gamma_X(h) = \gamma_Y(h) = \begin{cases} 3 & h = 0 \\ \sqrt{2} & h = 1 \\ 0 & h \geq 2 \end{cases}.$$



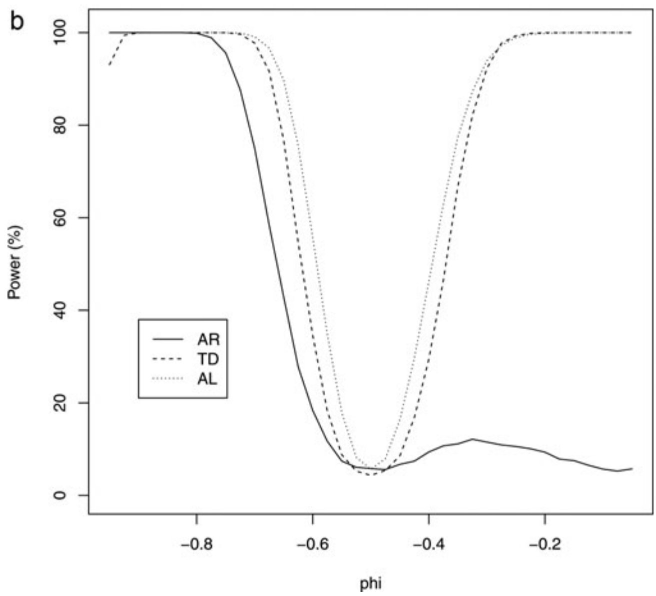


Figure 12. (a) Power when $\{X_t\}$ and $\{Y_t\}$ both follow an AR(1) model with $\phi_X = 0.5$ and $0.05 \leq \phi_Y \leq 0.95$. (b) Power when $\{X_t\}$ and $\{Y_t\}$ both follow an AR(1) model with $\phi_X = -0.5$ and $-0.95 \leq \phi_Y \leq -0.05$.

Figure 16 shows the power when θ_Y is allowed to range between 0 and 1.5. Both the time domain and bounded area tests increase power as θ_Y deviates from $1/\sqrt{2}$. The arc length test becomes erratic when $\theta_Y < 1/\sqrt{2}$.

Future research on this project will include trying to extend the bounded area test to stationary processes that are not independent from one another. For example, consider the

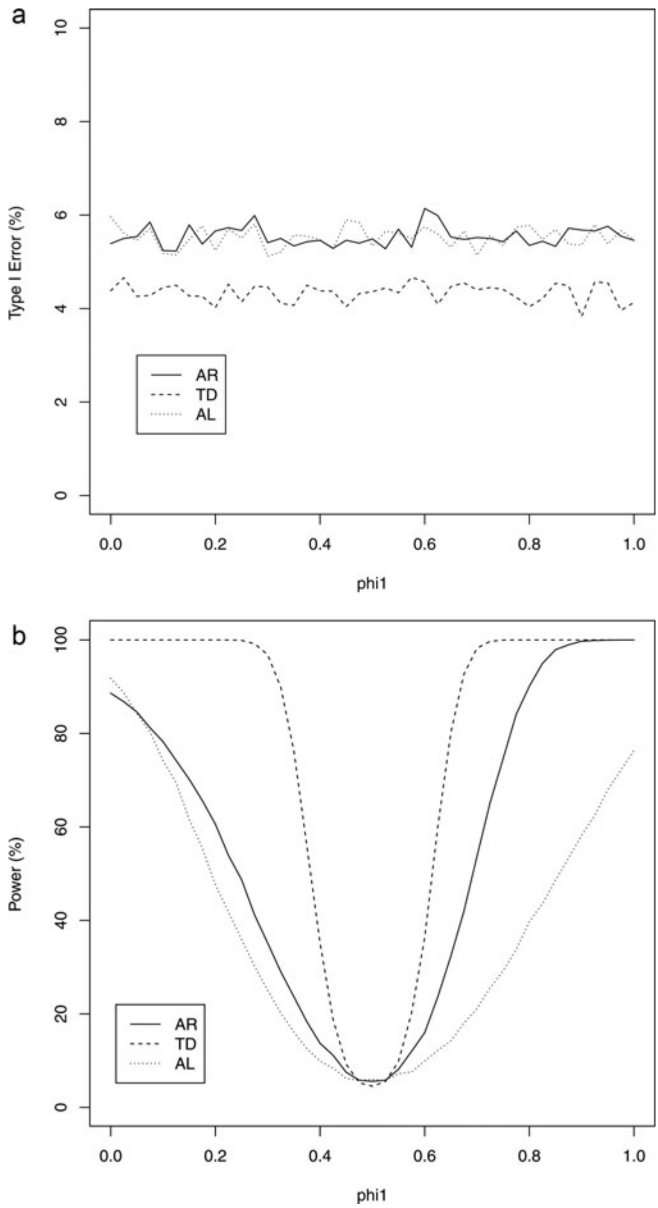


Figure 13. (a) Type I error when $\{X_t\}$ and $\{Y_t\}$ both follow an AR(2) model with $\phi_2^X = \phi_2^Y = -0.5$ and $0 \le \phi_1^X = \phi_1^Y \le 1$. (b) Power when $\{X_t\}$ and $\{Y_t\}$ both follow an AR(2) model with $\phi_2^X = \phi_2^Y = -0.5$, $\phi_1^X = 0.5$ and $0 \le \phi_1^Y \le 1$.

case where $\{Z_t\} \overset{\text{iid}}{\sim} N(0, 1)$ and $\{X_t\}$ and $\{Y_t\}$ are defined by

$$X_t = Z_t \qquad \text{and} \qquad Y_t = kX_t,$$

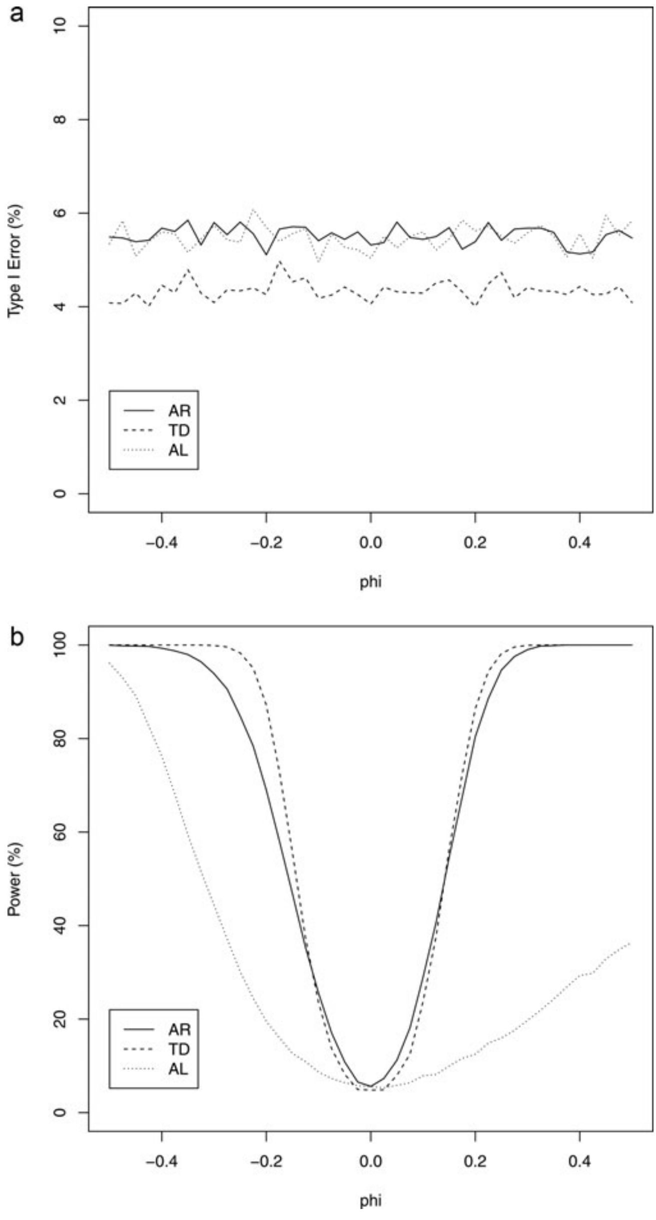


Figure 14. (a) Type I error when $\{X_t\}$ and $\{Y_t\}$ both follow an ARMA(1,1) model with $\theta_X = \theta_Y = 2$ and $-0.5 \leq \phi_X = \phi_Y \leq 0.5$. (b) Power when $\{X_t\}$ and $\{Y_t\}$ both follow an ARMA(1,1) model with $\theta_X = \theta_Y = 2$, $\phi_X = 0$ and $-0.5 \leq \phi_Y \leq 0.5$.

where k is a non zero constant. Then, we have

$$\gamma_Y(h) = k^2 \gamma_X(h) \qquad \text{and} \qquad \mathrm{Cov}(X_t, Y_t) = k.$$

Because $\{X_t\}$ and $\{Y_t\}$ are correlated, they are also dependent.

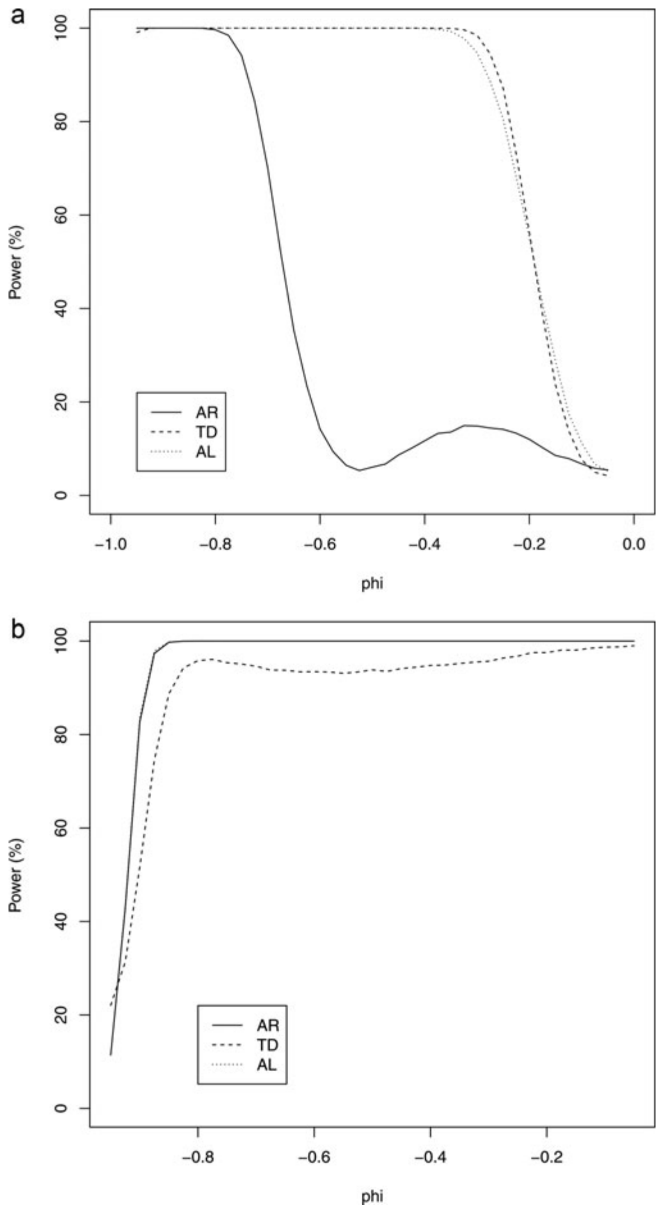


Figure 15. (a) Power when $\{X_t\}$ and $\{Y_t\}$ both follow an AR(1) model with $\phi_X = -0.05$ and $-0.95 \leq \phi_Y \leq -0.05$. (b) Power when $\{X_t\}$ and $\{Y_t\}$ both follow an AR(1) model with $\phi_X = -0.95$ and $-0.95 \leq \phi_Y \leq -0.05$.

Figure 17 shows the power when $k \in (0, 2]$. All three tests increase power rather quickly as k deviates from 1, despite the fact that the tests are treating the processes as if they are independent. It is doubtful, however, that ignoring dependence like this would be a successful tack in general. More properly, if $\{X_t\}$ and $\{Y_t\}$ are correlated, the test statistic

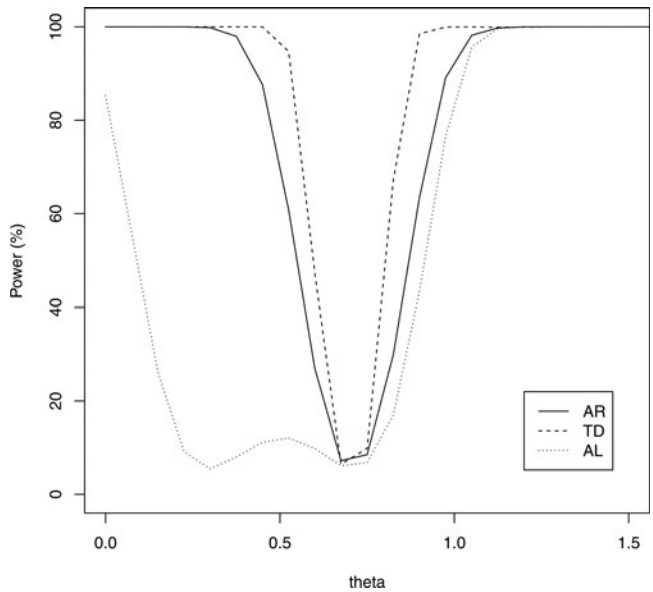


Figure 16. Power when $\{X_t\}$ and $\{Y_t\}$ both follow an MA(1) model with $\theta_X = \sqrt{2}$, $0 \leq \theta_Y \leq 1.5$, $\sigma_X^2 = 1$, and $\sigma_Y^2 = 2$.

in (2.4) should be rewritten to include a covariance term:

$$T \overset{H_0}{=} \frac{A_n^X - A_n^Y}{\sqrt{\text{Var}\left(A_n^X\right) + \text{Var}\left(A_n^Y\right) - 2\text{Cov}\left(A_n^X, A_n^Y\right)}}.$$

In practice, as long as consistent estimators are used for the terms in the denominator, Theorem 2.1 should still hold. Further investigations regarding dependence are currently underway.

A Computer Code

The bounded area formula given in (2.3) can be expressed in another way. For $t = 1, 2, \ldots, n$, define

$$\text{sign}(X_t) = \begin{cases} 1 & X_t > 0 \\ 0 & X_t = 0 \\ -1 & X_t < 0 \end{cases},$$

and for $t = 2, \ldots, n$, let

$$\eta_t = \begin{cases} 1 & \text{sign}(X_t) + \text{sign}(X_{t-1}) \neq 0 \\ 0 & \text{sign}(X_t) + \text{sign}(X_{t-1}) = 0 \end{cases} \qquad \text{and}$$

$$b_t = \begin{cases} t - \frac{X_t}{X_t - X_{t-1}} & X_t \neq X_{t-1} \\ 0 & X_t = X_{t-1} \end{cases}.$$

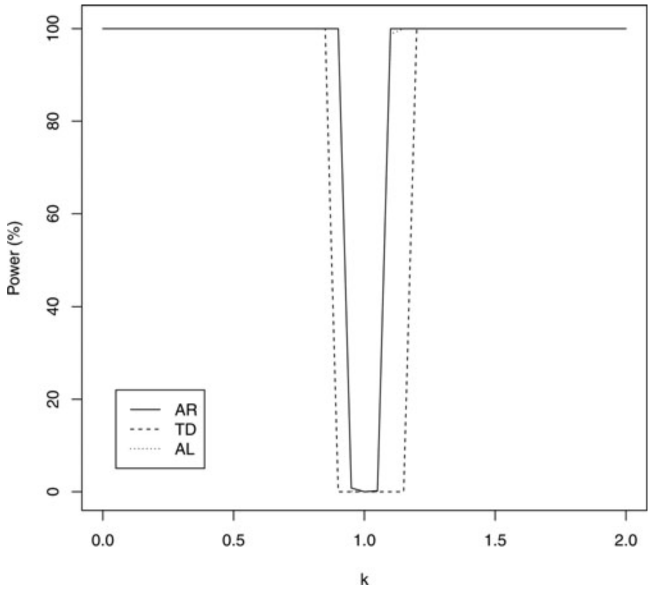


Figure 17. Power when $\{X_t\}$ and $\{Y_t\}$ are defined by $X_t = Z_t$ and $Y_t = kX_t$, where $\{Z_t\} \overset{\text{iid}}{\sim} N(0, 1)$ and $0 < k \leq 2$.

(2.3) can now be rewritten as

$$\sum_{t=2}^{n} \left[\frac{\eta_t}{2}(|X_{t-1}| + |X_t|) + \frac{1-\eta_t}{2}\left((t-b_t)|X_t| + (b_t - t + 1)|X_{t-1}|\right) \right].$$

While this new version is more complicated, it actually allows for an easy way to write computer code that can calculate the bounded area magnitude for sample $X_1, X_2, \ldots, X_n$. If this sample is contained in a vector called 'x' then the interested practitioner can use the following R code to get the area:

```
>  n=length(x)
>  a=1:(n-1)
>  for(j in 2:n){
+  if (x[j]==x[j-1]) b=0
+  else b=j-x[j]/(x[j]-x[j-1])
+  if (sign(x[j-1])+sign(x[j])==0)
+  a[j-1]=0.5*((b-j+1)*abs(x[j-1])+(j-b)*abs(x[j]))
+  else a[j-1]=0.5*(abs(x[j-1])+abs(x[j]))}
>  area=sum(a)
```

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