REVIEWS

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Probability Theory in Finance: A Mathematical Guide to the Black–Scholes Formula (Second edition). By Seán Dineen. American Mathematical Society, USA, 2013, xiv+305 pp., ISBN 978-0-8218-9490-3, $65.00 ($52.00 AMS members).

Reviewed by **Ferebee Tunno**

Back in 2011, I was researching the use of arc length as a measure of stock volatility. Having had a reasonable amount of experience with time series analysis, the stochastic nature of something like daily price changes was familiar territory, but financial mathematics was still somewhat of an unknown quantity. In an effort to become more savvy in such matters, I began perusing the literature and consequently found a number of texts that added to my general knowledge. Among these were [**1**], [**2**], and [**3**], but the one in particular that really captured my attention was *Probability Theory in Finance: A Mathematical Guide to the Black–Scholes Formula* by Seán Dineen. Over the next several months, I read various parts of the work, but ended up having to put it aside as the volatility investigations got superseded by other pressing projects. In 2013, however, I picked up the new second edition with a renewed interest (and a freer calendar) and set myself the strict goal of reading it cover to cover in a year and then writing a thorough review.

As the title suggests, the central purpose of the book is to derive, interpret, and utilize the Black–Scholes formula for pricing a call option. The work is based on a one-semester undergraduate course given to economics and finance students at University College Dublin. While a certain familiarity with the basic principles of finance is certainly required of the readership, Dineen has in actuality created a very sophisticated mathematical text. Specifically, its main topics include measure theory, conditional expectation, martingales, stochastic processes, Brownian motion, and the Itô integral. These topics are handled with great care and are gradually developed throughout the chapters without automatically assuming that the reader's background extends beyond integral calculus and differential equations.

The first two chapters start off the book by introducing some basic ideas about money and fair games. Specifically, Chapter 1 discusses risk and interest and their relationship to both present and discounted values of money. Chapter 2 discusses both fair and zero-sum games and what it means to calculate their expected winnings. By introducing the idea of expected winnings early on, Dineen is laying the groundwork for the more formal concept of expected value which is used extensively in later parts of the book.

Chapter 3 gets into σ-fields and how one can glean information from filtrations. Dineen's motive here is to show how an increasing sequence of nested fields can model the growing information associated with the passage of time, which is very important in the financial world. The main example he uses is the measurable space $(\Omega, \mathcal{F})$ where Ω denotes the set of all possible future prices that may be taken by a given

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share and $\mathcal{F}$ the set of events associated with the share prices. In this case, the natural filtration adapted to the share price is $(\mathcal{F}_n)_{n=1}^{\infty}$, where $\mathcal{F}_n$ is the history up to the nth day.

Chapters 4 and 5 discuss measurable functions and probability spaces. An $\mathcal{F}$-measurable function maps an event from σ-field $\mathcal{F}$ into a Borel set of real numbers. Such a transference allows one to measure the likelihood that such events occur. Thus, we now enter into the realm of probability and these $\mathcal{F}$-measurable functions are henceforth dubbed random variables. The financial tie-in arises when the first model for pricing a call option is constructed. Here we learn that a buyer can use the market price of the share (and interest rates) to determine a fair price for the option, while the seller will consider the price to be fair if it is possible to hedge any claim (i.e., reduce its risk). Independent events within a probability space are used to model the assumption that investors operate independently while buying and selling shares.

Chapters 6 and 7 delve into expected value, continuity, and integrability. Lebesgue integration is developed with respect to a probability measure and is defined successively for simple, positive bounded, positive, and then arbitrary random variables. Both the monotone and dominated convergence theorems are established in the process. The Riemann integral is also discussed and it is shown that the Lebesgue integral with respect to Lebesgue measure generalizes the Riemann integral. Riemann sums end up playing a role in defining the Itô integral later on in Chapter 12. The central limit theorem is stated as well, and is later used to help derive the Black–Scholes formula in Chapter 11.

Chapter 8 elaborates on and extends the first (binomial) model for pricing a call option first discussed in Chapter 5. Once again, a balanced portfolio to hedge any claim on the option is constructed, but now information is provided at (discrete) intermediate times during which trading is allowed. The concept of conditional expectation is introduced to be able to handle the new mathematics involved. Here Dineen is also setting the stage for martingales, which end up being essential for a more rigorous formulation of a set of fair games.

Chapter 9 delves deeper into Lebesgue measure, whose existence is formally proven using countable products of probability measures. The connection between Lebesgue and Riemann integration discussed in Chapter 7 is now put on a firmer foundation with the introduction of density functions, and it is shown that the former can in general handle a wider class of functions than the latter. Chapter 10 introduces martingales with examples including fair games, random walks, and Brownian motion. A martingale, in addition to allowing several important convergence results, is ultimately used to model the price of a call option.

In Chapter 11, we finally see the long-awaited Black–Scholes formula, named after economists Fisher Black (1938–1995) and Myron Scholes (b. 1941), and with important contributions by Robert Merton (b. 1944). The formula states that if the share price of a stock with volatility σ is X_0 today, then

$$X_0 N\left(\frac{\ln(X_0/k) + (r + 0.5\sigma^2)T}{\sigma\sqrt{T}}\right) - ke^{-rT} N\left(\frac{\ln(X_0/k) + (r - 0.5\sigma^2)T}{\sigma\sqrt{T}}\right)$$

is a fair price for a call option with maturity date T and strike price k, provided that r is the risk-free interest rate. Here $N(\cdot)$ is the distribution function for a standard normal random variable. Dineen initially arrives at the result using finite risk-neutral probabilities stemming from properties of elementary fair games, but he later presents another, equivalent derivation that calls upon properties of martingales and Brownian motion.

Chapter 12 brings things to a close by introducing stochastic integration, created by Kiyoshi Itô (1915–2008), as a way to hedge a call option. Itô integrals provide yet another way to reach the same price conclusion as the Black–Scholes formula.

Specifically, let V_t and $X_t = X_0 e^{\mu t + \sigma W_t}$ be the values of the option and share price, respectively, at time t, where $\{\mu t + \sigma W_t\}_{t \geq 0}$ is Brownian motion with drift. To hedge any claim on the option, a portfolio of θ_t shares and β_t units of a risk-free bond are held at time t. Supposing that one unit of the bond is worth $B(t)$ at time t, a hedge occurs if $dV_t = \theta_t dX_t + \beta_t dB(t)$. If $B(t) = e^{rt}$ with

$$\theta_t = N\left(\frac{\ln(X_t/k) + (r + 0.5\sigma^2)(T - t)}{\sigma\sqrt{T - t}}\right)$$

and

$$\beta_t = -ke^{-rT} N\left(\frac{\ln(X_t/k) + (r - 0.5\sigma^2)(T - t)}{\sigma\sqrt{T - t}}\right)$$

for $0 \leq t \leq T$, we have the same (asymptotic) result from the previous chapter.

So now that an overview of all the chapters has concluded, it's time to discuss the actual merits of Dineen's book. To begin, the content is developed on the appropriate high level and is complemented by both a clear writing style and a comfortable pace. Like any good mathematics text, every chapter builds upon its predecessors. No proposition is put forward that isn't used for some later purpose, and rarely is one presented without proof. All of the examples given serve their intended purpose of concept illustration.

One particular stylistic highlight is Dineen's use of anecdotal footnotes to give important results an historical setting. Nice biographical sketches are included for many mathematical "celebrities," among them Emile Borel, Paul Pierre Lévy, Joseph Leo Doob, and Guido Fubini. One minor notational improvement might be to use $\stackrel{\text{a.s.}}{\to}$ to denote almost sure convergence. This type of representation is already used elsewhere for convergence in distribution ($\stackrel{D}{\to}$) and $\mathbf{L}^p$ convergence ($\stackrel{\mathbf{L}^p}{\to}$), and since Dineen makes extensive use of sets that behave a certain way "almost everywhere," the use of $\stackrel{\text{a.s.}}{\to}$ would not only be consistent but also more efficient.

The second edition is not significantly different than the first, but there are a few changes. Chapters 1 through 8 are essentially the same in both editions, although some sections have been moved around and others have been combined for a more streamlined presentation. Specifically, sections 4.1 and 4.2 in the first edition (titled "The Borel Field" and "Measurable Functions," respectively) have been combined into Section 4.1 ("Measurable Functions") in the second edition. Section 7.1 of the first edition ("Summation of Series") has been moved to section 6.5 of the second, while section 7.7 ("Product Measures") has been moved to section 9.1 of the second. Chapter 9 of the second edition is completely new and is entitled "Lebesgue Measure." Chapters 9 through 11 of the first edition are now Chapters 10 through 12, respectively, of the second with very few changes.

There are exercises at the end of each chapter in both editions, but the second edition has added more problems to each set. Solutions or hints to most of these problems can be found at the back of the book. Overall, the second edition is an improvement upon the first edition, although the first was not substandard in any way. To be sure, there are still a few minor typographical errors in the second edition, but the reader can easily make his or her own quick corrections based on context.

For further comparable reading, I would recommend [**2**] and [**1**]. [**2**] is a more compact work and focuses mainly on stochastic differential equations. There is a nice build-up to the Black–Scholes formula as the feature application, but it doesn't go too deeply into areas like hedging and arbitrage. On the other hand, [**1**] is a more extensive work and delves into various topics that Dineen does not (e.g., a whole chapter is devoted to examining various rates of change of Black–Scholes option prices). Both texts are well written and serve as nice companion pieces to Dineen.

To conclude, I would certainly recommend Dineen's book to anyone with an interest in financial mathematics. Although the book was originally written with U.K. undergraduates in mind, I think it would work better here in the U.S. as a graduate-level text, and even then it is probably best spread out over two semesters. On the other hand, an advanced U.S. undergraduate could make an exception. Either way, Dineen has created a very nice work that has made a positive intellectual impact.

REFERENCES

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How to Make Equivalent Measures?

Let Σ be a σ-algebra and consider the finite measures μ and ν on Σ. The well-known Lebesgue decomposition theorem [**1**, Theorem 37.7] states that ν splits uniquely into $\nu_{a,\mu} \ll \mu$ and $\nu_{s,\mu} \perp \mu$. Furthermore, and this is the crucial point,

$$\nu_{a,\mu} = \sup_{n\in\mathbb{N}} \big(\nu \wedge (n\mu)\big) = \lim_{n\to\infty} \big(\nu \wedge (n\mu)\big),$$

$$\text{where} \quad \big(\nu \wedge (n\mu)\big)(S) = \inf_{T\in\Sigma} \big\{\nu(S\cap T) + n\mu(S\setminus T)\big\} \quad \text{for all} \quad S \in \Sigma.$$

Theorem. *Let μ and ν be two arbitrary finite measures on the σ-algebra Σ. Then the absolutely continuous parts with respect to each other are equivalent measures. That is, $\nu_{a,\mu} \ll \mu_{a,\nu}$ and $\mu_{a,\nu} \ll \nu_{a,\mu}$.*

Proof. Let $S \in \Sigma$ be a measurable set such that $\mu_{a,\nu}(S) = 0$. Observe that

$$0 = \mu_{a,\nu}(S) = \sup_{n\in\mathbb{N}} \big(\mu \wedge (n\nu)\big)(S) \geq \big(\mu \wedge (n\nu)\big)(S) \geq \big(\mu \wedge (\tfrac{1}{n}\nu)\big)(S) \geq 0$$

and therefore

$$0 = \big(\mu \wedge (\tfrac{1}{n}\nu)\big)(S) = n\big(\mu \wedge (\tfrac{1}{n}\nu)\big)(S) = \big((n\mu) \wedge \nu\big)(S)$$

holds for all $n \in \mathbb{N}$. This implies that $\nu_{a,\mu} \ll \mu_{a,\nu}$. Similarly, $\mu_{a,\nu} \ll \nu_{a,\mu}$. ■

So, to answer the question in the title: just do the Lebesgue decomposition.

REFERENCE

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—Submitted by Tamás Titkos

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